General Equations of Motion for an Elastic Wing and Method of Solution

Valter J. E. Stark*
Saab Scania AB, Linköping, Sweden

Equations in the time domain for arbitrary motion of a finite elastic wing in linearized flow are derived by means of more general indicial aerodynamic coefficients than those previously defined. Solution by the Laplace transformation shows that the normal response, for $M<1$, contains a line integral that depends on the discontinuity of the aerodynamic transfer functions across the negative real axis. For these functions, approximations of the Garrick type are utilized in a root-locus method in which no augmented states are needed. Indicial deficiency functions for a rectangular wing in incompressible flow are found to be almost equal for different deflection modes.

Nomenclature

A	= aspect ratio
$C^r_{m,n}(t)$	= indicial deficiency function
D	= system matrix determinant
$D^{n,m}$	= cofactor of element of D in the mth row and nth column
$D^r_{m,n}$	= apparent mass coefficient
$E_{n,m}(t)$	= exponential part of normal response $W_{n,m}(t)$
$H(t)$	= Heaviside's unit step function
$h_n(x,y)$	= given deflection mode referred to L
$I_{n,m}(t)$	= integral part of normal response $W_{n,m}(t)$
$K_{m,n}(t)$	= generalized aerodynamic coefficient referred to qS
$K^r_{m,n}(t)$	= indicial aerodynamic coefficient referred to qS
L	= reference length, e.g., mean semichord
$L\{f,p\}$	= Laplace transform of $f(t)$
$L^{-1}\{F,t\}$	= inverse Laplace transform of $F(p)$
$M_{m,n}$	= mass matrix element referred to m_r
m_r	= reference mass, e.g., wing mass
n_s	= number of terms of deflection approximation
p	= Laplace transform parameter referred to U/L
$\Delta p^r_n(t)$	= indicial pressure jump referred to q
$Q_{m,n}$	= system matrix element
q	= $\rho U^2/2$, dynamic pressure
$q_n(t)$	= generalized coordinate
S	= reference area, e.g., wing area
$S_{m,n}$	= stiffness matrix element referred to $m_r\omega_r^2$
t	= time referred to L/U
U	= freestream speed
v	= flight speed referred to $\omega_r L$
$W_{n,m}(t)$	= normal response or weighting function
x,y,z	= streamwise, spanwise, and normal coordinate referred to L
$\delta(t)$	= Dirac's delta function
μ	= $m_r/[(\pi/2)\rho SL]$, mass ratio
ρ	= freestream density
$\Phi(t)$	= Wagner's function
$\varphi^r_{m,n}$	= normalized indicial deficiency function
ϕ^r_n	= indicial perturbation velocity potential referred to UL
ω_r	= reference circular frequency, e.g., some natural circular frequency

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*Research Scientist. Member AIAA.

Introduction

USING indicial coefficients expressed in terms of the Wagner[1] function, Söhngen[2] derived an equation for arbitrary motion in the form of an integro-differential equation containing convolution integrals. In this paper, we shall derive corresponding equations of motion for a finite elastic wing in subsonic or supersonic linearized flow by using indicial coefficients appropriate for these cases. Strang,[3] Drischler,[4,5] Diederich and Drischler,[6] Vogel,[7] and Vepa[8] have given indicial coefficients for finite wings, but the indicial coefficients that are required for arbitrary deflection of an elastic finite wing have not been found in the literature.

We define two indicial coefficients for each deflection mode by considering two indicial solutions to the boundary value problem for the perturbation velocity potential of each mode. Aerodynamic coefficients for the arbitrary variation of the deflection with time are then written explicitly in terms of the generalized coordinates.

The initial value problem and the response problem can be solved by the Laplace transformation, which yields aerodynamic transfer functions corresponding to the indicial coefficients. As the transfer functions for finite aspect ratio A and nonzero Mach number M must approach the functions for $A=\infty$ and $M=0$ in a continuous way, the transfer functions for $M<1$ are discontinuous in a way similar to the Theodorsen function $C(-ip)$ on $\arg(p)=\pi$ (p being the transform parameter), i.e., on the negative real axis of the p plane.

Taking all of the singularities into account, we obtain the solution by reducing the inverse Laplace transform integral to integrals on small circles around the zeros of the system matrix determinant plus a line integral along the two sides of a cut through $\arg(p)=\pi$. The latter integral, which is significant for light wings, is nonzero for $M<1$ due to the discontinuity of the transfer functions, but it has no influence on the stability of the solution, which depends on the locations of the zeros—the eigenvalues. This was noted by Edwards.[9]

In order to determine the eigenvalues, it is necessary to approximate the transfer functions by simpler, but still accurate, functions. Expressions for this purpose have been published by Vepa,[8] Stark,[10] Edwards, Ashley, and Breakwell,[11] Edwards,[12] and Schwanz,[13] all of which are different from the expression proposed in this paper. Only this expression and that of Ref. 10 yield a discontinuity such as that of $C(-ip)$ on $\arg(p)=\pi$.

Padé approximations, which were proposed in Refs. 8, 11, and 12, and rational functions resulting from approximations of the Jones type to Wagner's function have poles instead of being discontinuous on the whole axis. Therefore, an approximation of the Garrick type[14] which has the proper discontinuity is proposed in this paper.

It has been found that two eigenvalues for the same flight speed very seldom coincide. Hence, it is possible and useful to apply the Newton-Raphson method as a basis for a program of root-locus generation. Combined with an efficient numerical routine for calculation of the determinant and its derivative, this method has been tested with good results.

Equations of Motion

Definition of Indicial Coefficients

Equations of motion that are valid for arbitrary variation of the wing deflection with time may be derived by means of indicial aerodynamic coefficients. Appropriate coefficients of this kind are derived here by considering indicial solutions to the boundary value problem for the perturbation velocity potential ϕ. Two indicial potentials ϕ_n^1 and ϕ_n^2 corresponding to two indicial normal velocities for each deflection mode are required.

Using dimensionless coordinates x, y, and z (referred to some reference length L), the wing deflection is described by the equation

$$z=\sum_{n=1}^{n_s} h_n(x,y)q_n(t) \tag{1}$$

where $h_n(x,y)$ is a given deflection mode (a function of the coordinates x and y in the wing plane) and $q_n(t)$ a generalized coordinate (a continuous but otherwise arbitrary undetermined function of the dimensionless time t which is referred to L/U, U being the freestream speed). The corresponding indicial aerodynamic coefficients are defined by

$$K_{m,n}^r(t)=\frac{1}{S}\iint h_m \Delta p_n^r \mathrm{d}S \qquad r=1 \text{ or } 2 \tag{2}$$

The integration region is the wing projection, S some reference area, $\mathrm{d}S$ the surface element, and Δp_n^r the dimensionless indicial pressure jump (in the z direction) that corresponds to the indicial potential ϕ_n^r. The linearized Bernoulli equation yields

$$\Delta p_n^r=-2\left(\frac{\partial \Delta\phi_n^r}{\partial x}+\frac{\partial \Delta\phi_n^r}{\partial t}\right) \tag{3}$$

where Δp_n^r and $\Delta\phi_n^r$ are referred to the freestream dynamic pressure $\rho U^2/2$ and UL, respectively.

Like the ordinary perturbation velocity potential, the indicial potential satisfies the usual linearized partial differential equation (the wave equation referred to moving coordinates) and the usual conditions, which include the radiation condition at distant points, the Kutta condition at the trailing edge, the condition of zero pressure jump across the wake, and a condition for the normal velocity component. Defining two indicial solutions for each mode, we demand that, on the wing,

$$\begin{aligned}\frac{\partial \phi_n^r}{\partial z}&=\left(\frac{\partial h_n}{\partial x}\right)H(t) \quad &\text{for } r=1\\ &=h_n H(t) \quad &\text{for } r=2\end{aligned} \tag{4}$$

where $H(t)$ is the Heaviside unit step function.

The desired aerodynamic coefficient for arbitrary deflection is defined by

$$K_{m,n}(t)=\frac{1}{S}\iint h_m \Delta p_n \mathrm{d}S \tag{5}$$

where $\Delta p_n=-2(\partial\Delta\phi_n/\partial x+\partial\Delta\phi_n/\partial t)$. The potential ϕ_n, which corresponds to the nth term of Eq. (1), satisfies the ordinary condition for the normal velocity on the wing

$$\frac{\partial \phi_n}{\partial z}=\left(\frac{\partial h_n}{\partial x}\right)q_n+h_n\dot{q}_n \tag{6}$$

where $\dot{q}_n=\mathrm{d}q_n/\mathrm{d}t$.

Aerodynamic Forces for General Motion

It is required to express the coefficient $K_{m,n}(t)$ explicitly in terms of the generalized coordinates. Since all of the relations involved are linear, this can be achieved by superposition in the same way as in the theory of linear filters[15]; the value of the response at time t to an arbitrary input is given by a convolution integral with the integrand formed by the unit step response for the argument $t-\tau$ multiplied by the derivative of the input at time τ.

In the present case, the input consists of both q_n and $\dot{q}_n$, but these can be treated separately and the responses added. They have to be treated separately since they are multiplied by different functions, as seen from Eq. (6). For this reason, $K_{m,n}^1(t)$ and $K_{m,n}^2(t)$ must be assumed to be different in the general case. Assuming that $q_n(t)$ is identically zero for $t<0$, we find in the way indicated that

$$K_{m,n}(t)=\int_0^t K_{m,n}^1(t-\tau)\dot{q}_n(\tau)\mathrm{d}\tau+\int_{0-}^t K_{m,n}^2(t-\tau)\ddot{q}_n(\tau)\mathrm{d}\tau \tag{7}$$

This relation, which represents the desired result, is useful even if $\dot{q}_n(t)$ is discontinuous, but the second integral must be evaluated accordingly. For example, if $\dot{q}_n(t)$ suddenly assumes a finite value $\dot{q}_n(0)$ at $t=0$, the integral receives the contribution $K_{m,n}^2(t)\dot{q}_n(0)$ from a vanishing part of the integration region around $t=0$.

It seems useful to decompose the indicial coefficients as

$$K_{m,n}^r(t)=K_{m,n}^r(\infty)-C_{m,n}^r(t)+D_{m,n}^r\delta(t) \tag{8}$$

where $\delta(t)$ is the Dirac delta function. This function is involved only for incompressible flow, in which case $D_{m,n}^r$ is the apparent mass coefficient. The function $C_{m,n}^r(t)$ may be called the deficiency function, since it represents the difference between the steady-state limit $K_{m,n}^r(\infty)$ and $K_{m,n}^r(t)$. Using this decomposition, we get

$$\begin{aligned}K_{m,n}(t)=&K_{m,n}^1(\infty)q_n+[K_{m,n}^2(\infty)+D_{m,n}^1]\dot{q}_n+D_{m,n}^2\ddot{q}_n\\ &-\int_0^t C_{m,n}^1(t-\tau)\dot{q}_n(\tau)\mathrm{d}\tau-\int_{0-}^t C_{m,n}^2(t-\tau)\ddot{q}_n(\tau)\mathrm{d}\tau\end{aligned} \tag{9}$$

Since the limit of $C_{m,n}^r(t)$ for $t\to\infty$ is zero, the integrals in this expression are handled more easily than those in Eq. (7).

Equations for General Motion

Having obtained general expressions for the aerodynamic coefficients $K_{m,n}(t)$, the desired equations of motion are easy to find. Using Lagrange's equations and disregarding the structural damping forces, we write them in the dimensionless form

$$\sum_{n=1}^{n_s}[M_{m,n}v^2\ddot{q}_n+S_{m,n}q_n+(v^2/\pi\mu)K_{m,n}(t)]=f_m(t) \qquad m=1(1)n_s \tag{10}$$

where $f_m(t)$ is a generalized exciting force coefficient and $M_{m,n}$ and $S_{m,n}$ the normalized mass and stiffness matrix elements, respectively. The quantities $v = U/(\omega_r L)$ and $\mu = m_r/[(\pi/2)\rho LS]$ are dimensionless speed and mass parameters defined by means of a typical circular frequency ω_r and a reference mass m_r (e.g., a natural circular frequency and the corresponding generalized mass).

Since $K_{m,n}(t)$ contains convolution integrals, Eqs. (10) may be called integro-differential equations. If the right-hand members $f_m(t)$ are zero, which is the case in the flutter problem, the equations are homogeneous.

Solution of the Equations

Laplace Transformation

Contrary to ordinary homogeneous differential equations, the homogeneous integro-differential equations do not have a solution of the form e^{pt} for any value of p for $M<1$. But functions of this kind form an important part of the complete solution, which can be obtained with the Laplace transformation.

The Laplace transform $L\{f,p\}$ of a function $f(t)$ is defined by the relation

$$L\{f,p\} = \int_0^\infty e^{-pt} f(t)\,\mathrm{d}t \tag{11}$$

for those values of the transform parameter p for which the integral exists and for other values by analytic continuation of the function that is defined by the integral. If $f(t)$ is such that the integral can be evaluated in closed form, this continuation is immediately obtained. For example:

$$L\{1,p\} = 1/p \quad \text{for } p \neq 0$$

$$L\{e^{at},p\} = 1/(p-a) \quad \text{for } p \neq a$$

$$L\{(a+t)^{-1},p\} = e^{ap} E_1(ap) \quad \text{for } |\arg(ap)| < \pi \ \ (\text{and } a \neq 0)$$

$E_1(z)$ being the exponential integral.

According to the inversion theorem, the inverse Laplace transform $L^{-1}\{F,t\}$ of a transform $F(p)$ is given by the line integral

$$L^{-1}\{F,p\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} L\{f,p\}\,\mathrm{d}p \tag{12}$$

where the integration path $p = c$ has such a location that $L\{f,p\}$ exists and is analytic for $Re(p) \geq c$. However, by means of Cauchy's theorem this path can be replaced by any path provided $F(p) = L\{f,p\}$ is analytic at all points on and to the right of the path.

Paying particular attention to the convolution integrals, we find that Laplace transformation of Eq. (9) for both subsonic and supersonic flow gives

$$L\{K_{m,n},p\} = A_{m,n}(p) L\{q_n,p\} - D^2_{m,n} \dot{q}_n(0) \tag{13}$$

where

$$A_{m,n}(p) = K^1_{m,n}(\infty) + [K^2_{m,n}(\infty) + D^1_{m,n}]p + D^2_{m,n} p^2$$
$$-p[L\{C^1_{m,n},p\} + pL\{C^2_{m,n},p\}] \tag{14}$$

The quantities $A_{m,n}(p)$ may be called aerodynamic transfer functions, since they correspond to the transfer function in the linear filter theory.[15] They must agree with the unsteady aerodynamic coefficients that are defined by oscillating-surface theories. Since the imaginary unit enters only via p, the two functions must agree for arbitrary complex values of p ($p = ik$, k being the reduced frequency).

Laplace transformation is now applied to each member of the equations of motion (10), which transform into

$$\sum_{n=1}^{n_s} Q_{m,n} L\{q_n,p\} = c_m + L\{f_m,p\}, \qquad m = 1(1)n_s \tag{15}$$

where

$$Q_{m,n} = M_{m,n}(vp)^2 + S_{m,n} + (v^2/\pi\mu) A_{m,n}(p) \tag{16}$$

and

$$c_m = v^2 \sum_{n=1}^{n_s} \left[M_{m,n} + \left(\frac{1}{\pi\mu}\right) D^2_{m,n} \right] \dot{q}_n(0) \tag{17}$$

Solution

Using the determinant $D = D(p)$ of the system matrix $[Q_{m,n}]$ and the cofactor $D^{n,m} = D^{n,m}(p)$ of the element in the mth row and the nth column of the matrix, we may express the inverse transform of the solution to the algebraic equations (15) as

$$q_n(t) = \sum_{m=1}^{n_s} \left[c_m L^{-1}\{F,t\} + \int_0^t L^{-1}\{F,t-\tau\} f_m(\tau)\,\mathrm{d}\tau \right]$$
$$n = 1(1)n_s \tag{18}$$

where $F = F(p) = D^{n,m}/D$. The evaluation of the inverse transform $L^{-1}\{F,t\}$ is facilitated by choosing a suitable integration path, which in turn implies that the singularities of $F(p)$ must be considered.

In order that $A_{m,n}(p)$ shall approach the special transfer functions for incompressible and two-dimensional flows in a continuous way, it is necessary that $A_{m,n}(p)$ for $M<1$ be discontinuous on $\arg(p) = \pi$ in the same way as the special functions. For $M>1$, the deficiency functions are identically zero after a certain time which seems to imply that $A_{m,n}(p)$ is analytic for all finite values of p in that case. The remaining singularities are zeros of D, $p = p_\nu$. Keeping in mind that all singularities must stay to the left of the path, we see that this may be reduced to the path employed by Sears.[16] This consists of small circles around the zeros, plus two lines closely following the negative real axis (one on each side), plus a small circular arc around the origin, plus two quarter-circles in the left half-plane with large radii and center the origin.

The integral over each quarter-circle tends to zero if the radius R of the circle is large enough. In order to show this, we note that D and $D^{n,m}$ are sums of products with different numbers of factors; the products for D contain n_s factors, but those for $D^{n,m}$ only $n_s - 1$ factors.

For incompressible flow, $|A_{m,n}(p)|$ and $|Q_{m,n}|$ behave like $|p|^2$ for $|p| \to \infty$. Due to the different numbers of factors, $|pF(p)|$ is therefore less than ϵ at all points of the quarter-circle, so that the integral of this is less than some constant times ϵ for R large enough.

For compressible flow, $|A_{m,n}(p)|$ may be assumed to increase either more or less rapidly than $|p|^2$. However, $|Q_{m,n}|$ increases at least as rapidly as $|p|^2$ in both cases, so that $|pF(p)|$ becomes less than ϵ at all points on the quarter-circle for $M>0$. Hence, the contributions from the quarter-circles tend to zero for $R \to \infty$ for both $M<1$ and $M>1$.

With regard to the zeros of D, it is noted that the analytic function D has real structure, which means that the imaginary unit enters only via p. For this reason, the complex conjugate $\bar{p}_\nu$ of a zero is also a zero.

The number of zeros or roots of the equation $D = 0$ must be determined. In the absence of aerodynamic forces, the number of pairs of roots $(p_\nu, \bar{p}_\nu)$ is exactly n_s [at least if the deflection modes $h_n(x,y)$ are natural modes (ground vibration modes)]. For low speeds, the number of roots no doubt remains the same; even for higher speeds, it appears unlikely that more than n_s relevant pairs of roots should exist. The actual number can be determined by "the principle of the argument" which is proved in the theory of analytic functions.

Assuming that the zeros are simple zeros, we may approximate D in the vicinity of a zero by the linear term $(p-p_\nu)D'(p_\nu)$ of its Taylor series expansion about p_ν. If this term is used for approximation of the integrand on the small circles around the zeros, it is seen that part of the inverse transform $L^{-1}\{F,t\}$ is given by

$$E_{n,m}(t) = \sum_{\nu=1}^{\nu_S} \left[e^{p_\nu t} \frac{D^{n,m}(p_\nu)}{D'(p_\nu)} + e^{\bar{p}_\nu t} \frac{D^{n,m}(\bar{p}_\nu)}{D'(\bar{p}_\nu)} \right] \quad (19)$$

where $\nu_s = n_s$ probably. This may be called the exponential part. The remaining part, which may be called the integral part since it results from the integrals on the lines along the negative real axis, can be reduced to

$$I_{n,m} = \frac{-1}{2\pi i} \int_{-\infty+i0+}^{0+i0+} e^{pt} [F(p) - F(\bar{p})] \mathrm{d}p \quad (20)$$

where the integration path follows the upper side of the negative real axis. For $M<1$, this integral is in general not zero because $F(p)$ is different from $F(\bar{p})$ on the whole path.

The two parts are seen to be real, as they should be. Added together, they yield the complete inverse transform

$$L^{-1}\{F,p\} = W_{n,m}(t) = E_{n,m}(t) + I_{n,m}(t) \quad (21)$$

Inserting this in Eq. (18), we get the final result

$$q_n(t) = \sum_{m=1}^{n_S} [c_m W_{n,m}(t) + \int_0^t W_{n,m}(t-\tau) f_m(\tau) \mathrm{d}\tau] \qquad n = 1(1)n_s \quad (22)$$

The function $W_{n,m}(t)$ may be called normal response or weighting function, since it corresponds to a function with those names in linear filter theory.[15] In that theory, the weighting function is composed of functions all of which have the form $e^{p_\nu t}$ (or $t^k e^{p_\nu t}$), but this is not the case here due to the presence of the integral part.

For values of v such that there is no root on the negative real axis, the integrand part within brackets in Eq. (20) is numerically less than some positive number so that $|I_{n,m}(t)|$ is less than some positive number divided by t for $t \to \infty$. Therefore, unstable contributions to the solution can originate only from the exponential part. They correspond to roots in the right half-plane.

The size of the contribution $I_{n,m}$ has been studied in a simple example of rigid wing performing a plunging motion with given initial velocity in two-dimensional incompressible flow. It was found that $I_{n,m}$ was relatively small for $\mu = 13$, but large for $\mu = 2$. In this example, m_r is the wing mass, S the wing area, and L the semichord.

Approximation of Indicial Coefficients

Expressions for Deficiency Functions

According to Eq. (14), the aerodynamic transfer functions $A_{m,n}(p)$ are determined by the Laplace transforms of the deficiency functions $C^r_{m,n}(t)$. Hence, expressions for the approximation of the former can be obtained by transformation of expressions for the latter.

For $M=0$ and two-dimensional flow, it is known that the Garrick approximation[14] $[1-2/(4+t)]$ of the Wagner function $\Phi(t)$ yields better than 2% accuracy. The Jones approximation, which has the form $(1-A^{-\alpha t}-B^{-\beta t})$ with α and β positive, is also accurate, but it approaches unity too rapidly as $t \to \infty$. It yields approximations similar to the Padé approximations[8] in the frequency domain (i.e., in the p plane).

The Garrick approximation has the proper behavior for $t \to \infty$, which implies that a corresponding approximation to the Theodorsen function $C(-ip)$, which results from Laplace transformation, will receive the proper behavior. The function obtained in the frequency domain via the Garrick approximation contains a logarithmic term which yields a distributed discontinuity like that of $C(-ip)$ on the negative real axis of the p plane. For this reason, we prefer an approximation of the Garrick type instead of the Jones approximation or Padé approximations which have poles on the negative real axis.

For $M<1$ and two- or three-dimensional flow, we propose an expression of the form

$$C^r_{m,n}(t) = \sum_{k=k_i}^{k_S} B^{r,k}_{m,n} \left(\frac{a}{a+t}\right)^k \quad (23)$$

This is attractive, since for $M=0$ it may suffice to retain only one term. The choice $k_i = k_s = 1$, $a=4$ yields, namely, the Garrick approximation, and $k_i = k_s = 3$, $a = 5.5$ yields an accurate approximation for a rectangular wing, which is shown in a later section. Diagrams in Ref. 17 indicate that $K^r_{m,n}(t)$ for $M>0$ may have a peak at $t=0$ instead of the delta function, but such a peak can be represented by one or a combination of a few of the functions included in Eq. (23). Also, $C^r_{m,n}(t)$ is a regular function of the variable $s = a/(a+t)$ in the interval $0<s<1$. Equation (23), which is a polynomial in s, should yield a close approximation to such a function if the parameters k_i, k_s, and a and the constants $B^{r,k}_{m,n}$ are given suitable values.

For $M>1$, the indicial coefficients reach their steady-state values after a finite time, at $t=a$ say, and the deficiency functions are at least continuous in the interval $0 \le t \le a$. Therefore and for simplicity, an expression of the form

$$C^r_{m,n}(t) = \sum_{k=k_i}^{k_S} B^{r,k}_{m,n} \left(1-\frac{t}{a}\right)^k \qquad 0 \le t \le a$$

$$= 0 \qquad t>a \quad (24)$$

is proposed in this case.

Indicial Coefficients for a Rectangular Wing

The method of Belotserkovskii,[18] which is a vortex method applicable to finite wings and arbitrary variation of the normal velocity in space and time, can easily be programmed for a rectangular wing and $M=0$. As an earlier application showed that this method could produce very accurate results for the Wagner function, it has been employed for the calculation of indicial coefficients for a rectangular wing of aspect ratio $A=3$.

On the chord 20 equidistant spanwise vortices were used and on the wake up to 100, which correspond to $t=10$; the reference length L was chosen equal to the semichord. Equidistant chordwise vortices were also used with a total number of 10.

The first spanwise vortex on the wing, the first spanwise vortex on the wake, and the chordwise vortices at the side edges were placed at a certain distance from the leading, trailing, and side edges, respectively. In order to obtain accurate results, this distance was chosen equal to one-quarter of the distance between the spanwise vortices in the first two cases and equal to one-quarter of the distance between the chordwise vortices in the last cases.

The calculation was performed for four typical deflection modes defined by

$$[h_n(x,y)] = [1, g_1, x-x_m, (x-x_m)g_2] \tag{25}$$

where x_m is the x coordinate at the 50% chord line, and the streamwise symmetry axis is the x axis. For simplicity, we let the bending and torsion functions be equal and defined by $g_1 = g_2 = 1.2\eta^2 - 0.2\eta^4$ where $\eta = y/A$.

The corresponding indicial normal velocities may be written

$$\left[\frac{\partial \phi_n^1}{\partial z}\right] = [0, 0, 1, g_2] H(t) \tag{26}$$

and

$$\left[\frac{\partial \phi_n^2}{\partial z}\right] = [1, g_1, x-x_m, (x-x_m)g_2] H(t) \tag{27}$$

In this example, where $g_2 = g_1$, it is sufficient to consider the velocities $\partial\phi_n^2/\partial z$, since, according to Eqs. (26) and (27), indicial coefficients corresponding to $\partial\phi_n^1/\partial z$ are zero for $n=1$ and 2 and identical to those for $\partial\phi_{n-2}^2/\partial z$ for $n=3$ and 4.

The exact values of some of the quantities $D_{m,n}^2$, $K_{m,n}^2(0+)$, and $K_{m,n}^2(\infty)$ are known in the case of $A=\infty$. These values are

$$D_{1,1}^2 = \pi, \quad D_{1,3}^2 = 0, \quad D_{3,1}^2 = 0, \quad D_{3,3}^2 = \pi/8$$

$$K_{1,1}^2(0+) = \pi, \quad K_{1,3}^2(0+) = \pi/2, \quad K_{3,1}^2(0+) = -\pi/2$$

$$K_{3,3}^2(0+) = \pi/4, \quad K_{1,1}^2(\infty) = 2\pi, \quad K_{1,3}^2(\infty) = \pi$$

$$K_{3,1}^2(\infty) = -\pi, \quad K_{3,3}^2(\infty) = 0$$

The quantities have also been calculated by the vortex method, but of course without using streamwise vortices. The results are given in Tables 1-4, and comparison with the exact values shows that there is close agreement. S is the wing area.

Normalized Deficiency Function

The results for the function $C_{m,n}^r(t)$ shall be described by considering the normalized deficiency function

$$\varphi_{m,n}^r(t) = C_{m,n}^r(t)/C_{m,n}^r(0) \tag{28}$$

This is considered because there is reason to believe that $C_{m,n}^r(t)$ varies in almost the same way with t for all deflection modes. If this is true, the normalized deficiency functions should be almost identical.

In order to investigate this possibility, Eq. (28) has been calculated for $t = 0(0.1)10$ by using data from the vortex method. By plotting the results for different m and n in the same diagram, it was found that all points fell very close to a single curve, which is the curve labeled $A=3$ in Fig. 1; the deviations are hardly greater than the width of the curve.

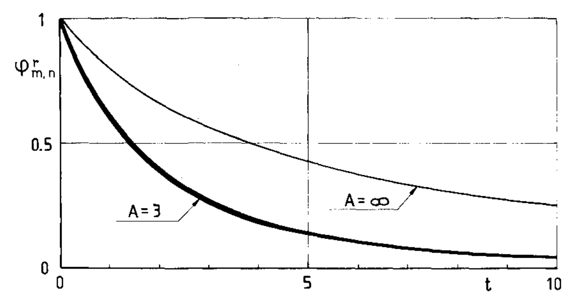

Fig. 1 Normalized deficiency functions for a rectangular wing in incompressible flow.

Due to this result, we are tempted to hypothesize that $\varphi_{m,n}^r(t)$ for a rectangular wing in incompressible flow is independent of the deflection mode. Such a hypothesis is supported by the theory for two-dimensional incompressible flow, which shows that

$$\varphi_{m,n}^r(t) = 2[1-\Phi(t)] \tag{29}$$

for all modes (plunge, pitch, and control surface deflection). Equation (29) yields the curve for $A=\infty$ in Fig. 1. Although a closer investigation will probably show that the hypothesis

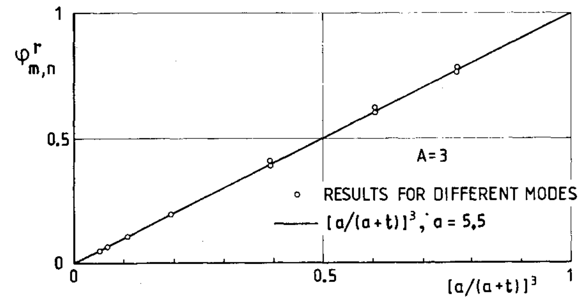

Fig. 2 Approximation of normalized deficiency functions for a rectangular wing.

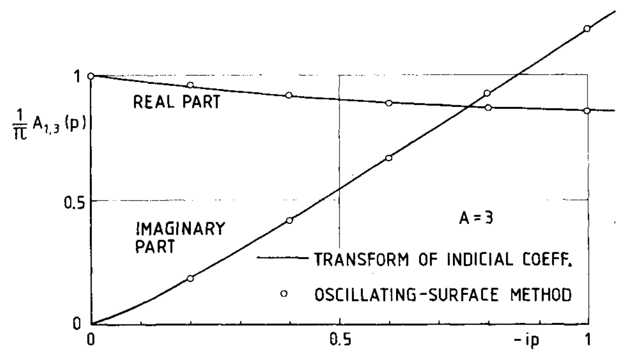

Fig. 3 Comparison of results for lift due to pitch from the indicial coefficient and oscillating surface methods.

Table 1 Results for $D_{m,n}^2$

A	m \ n	1	2	3	4
3	1	2.579	0.762	0.	0.
	2	0.763	0.358	0.	0.
	3	0.	0.	0.360	0.114
	4	0.	0.	0.113	0.061
∞	1	3.14	—	0.	—
	2	—	—	—	—
	3	0.	—	0.39	—
	4	—	—	—	—

Table 2 Results for $K_{m,n}^2(0+)$

A	m \ n	1	2	3	4
3	1	2.357	0.672	1.297	0.384
	2	0.670	0.284	0.384	0.181
	3	−1.297	−0.384	0.711	0.222
	4	−0.384	−0.176	0.221	0.116
∞	1	3.14	—	1.57	—
	2	—	—	—	—
	3	−1.57	—	0.78	—
	4	—	—	—	—

cannot be generally valid (for special deflection modes the flow can be almost two-dimensional for any aspect ratio), it is possible that it can be accepted for practical flutter calculations.

The results obtained from the vortex method can also be used for checking the usefulness of the approximating equation (23). The straight line in Fig. 2 represents a function included in this expression, while the circles represent a few of the values calculated by the vortex method. Since only one term has been employed and since in spite of this the agreement is very good, the check confirms that Eq. (23) is useful, at least for rectangular wings in incompressible flow.

Approximation of Transfer Functions

Expression for Transfer Functions

Laplace transformation of Eq. (23) or (24) and insertion of the results into Eq. (14) gives

$$A_{m,n}(p) = K^1_{m,n}(\infty) + [K^2_{m,n}(\infty) + D^1_{m,n}]p + D^2_{m,n}p^2 - \sum_{k=k_i}^{k_s} (B^{1,k}_{m,n} + pB^{2,k}_{m,n})apF_k(ap) \qquad (30)$$

where

$$F_k(p) = \int_0^\infty e^{-pt}(1+t)^{-k}dt = \frac{1-pF_{k-1}(p)}{k-1} \qquad M<1$$

$$= \int_0^1 e^{-pt}(1-t)^k dt = \frac{1-kF_{k-1}(p)}{p} \qquad M>1 \qquad (31)$$

and $F_1(p) = e^p E_1(p)$ for $M<1$ and $F_0(p) = (1-e^{-p})/p$ for $M>1$. The function $E_1(p)$ is the exponential integral for which Gautschi and Cahill[19] give the expansion

$$E_1(p) = -\gamma - \ln(p) - \sum_{n=1}^{\infty} \frac{(-1)^n p^n}{n(n!)} \qquad |\arg(p)| < \pi \qquad (32)$$

where γ is Euler's constant, 0.577215....

$F_k(p)$ for $M<1$ is analytic for $|\arg(p)|<\pi$, and $F_k(p)$ for $M>1$ is analytic for all finite values of p.

Comparison of Results for Transfer Functions

From Eq. (30) and the check described above it follows that aerodynamic transfer functions for a rectangular wing in incompressible flow may be written approximately as

$$A_{m,n}(p) = K^1_{m,n}(\infty) + [K^2_{m,n}(\infty) + D^1_{m,n}]p + D^2_{m,n}p^2 - [C^1_{m,n}(0) + pC^2_{m,n}(0)]apF_3(ap) \qquad (33)$$

The constants $K^r_{m,n}(\infty)$, $D^r_{m,n}$, and $C^r_{m,n}(0)$, as calculated by the vortex method for $A=3$, are given for $r=2$ in Tables 1-4 and for $r=1$ by those for $r=2$ as described. The result of using these values and $a=5.5$ in an evaluation of Eq. (33) for imaginary values of p is shown by the solid curves in Fig. 3 for $m=1$ and $n=3$.

It is interesting to compare these results with coefficients calculated by an oscillating surface method. For such a comparison we use data published by Stark,[20] but first we have to transform these data from a reference axis at 25% chord to an axis at 50%. As the resulting values, which are represented by the circles in Fig. 3, are seen to be in very good agreement with those calculated via indicial coefficients, both methods are considered accurate. But the indicial method is preferable, since it yields accurate values for all complex values of p.

Approximation in Frequency Domain

Oscillating surface programs are usually available, while programs for calculation of indicial coefficients are not. Therefore, it is desirable to utilize the former programs for calculation of $A_{m,n}(p)$ for discrete values of p and to determine approximations on the basis of data calculated in this way.

As Eq. (30) results from expressions considered suitable for indicial coefficients, it should be appropriate to choose this expression for approximation of $A_{m,n}(p)$. The constants of the expression, which are now considered arbitrary, may be determined by the method of least squares. The apparent mass coefficients $D^1_{m,n}$ and $D^2_{m,n}$ shall be set equal to zero if $M>0$.

As an example, Eq. (33), which corresponds to Eq. (30), has been determined by using data from Ref. 20. The constants were determined on the basis of the six values shown by circles in Fig. 3 by the method of least squares. The results obtained for the constants were then inserted in the expression and this was calculated for complex values of p. The expression was also calculated for complex p values by using data for the constants from the vortex method, i.e., from Tables 1-4. The results from the two calculations are compared in Figs. 4 and 5.

The results shown in Fig. 3 from the oscillating surface method (the circles) are very close to those calculated via indicial coefficients and the vortex method (the solid lines), but there are slight differences. Although slight, these differences may be significant. They are the cause of a somewhat greater difference for complex values of p, which appears primarily in a region around the negative real axis as is shown in Figs. 4 and 5. Thus, it seems important to employ an oscillating surface method that yields accurate results.

A Root-Locus Method

The Newton-Raphson formula $p_2 = p_1 - D(p_1)/D'(p_1)$ for finding a more accurate value p_2 than a given approximate value p_1 for a root of a nonlinear equation $D(p)=0$ is applicable in the present case, since the determinant $D(p)$ is analytic. But as it is not feasible to form explicit analytic expressions for $D(p)$ and the derivative $D'(p)$, a root-locus program has been developed by using a finite difference formula for calculation of the derivative and a modified

Table 3 Results for $K^2_{m,n}(\infty)$

A	m^n	1	2	3	4
3	1	3.133	0.870	1.720	0.495
	2	0.870	0.340	0.493	0.211
	3	−1.720	−0.493	0.480	0.161
	4	−0.495	0.211	0.161	0.099
∞	1	6.27	—	3.14	—
	2	—	—	—	—
	3	−3.14	—	0.	—
	4	—	—	—	—

Table 4 Results for $C^2_{m,n}(0)$

A	m^n	1	2	3	4
3	1	0.776	0.199	0.423	0.111
	2	0.200	0.056	0.109	0.030
	3	−0.423	−0.109	−0.231	−0.061
	4	−0.111	−0.035	−0.060	−0.017
∞	1	3.13	—	1.57	—
	2	—	—	—	—
	3	−1.57	—	−0.78	—
	4	—	—	—	—

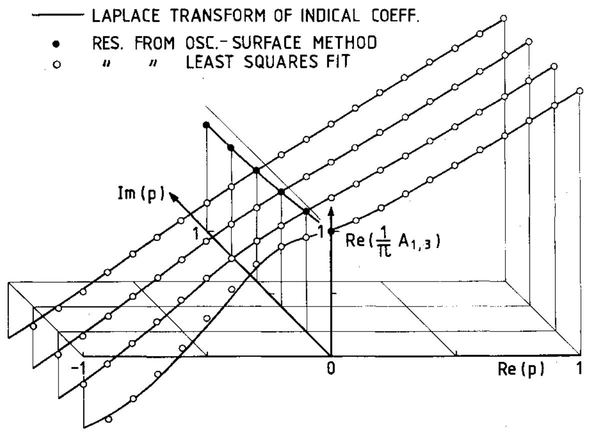

Fig. 4 Comparison of real parts of transfer functions from indicial coefficient method and least squares fit.

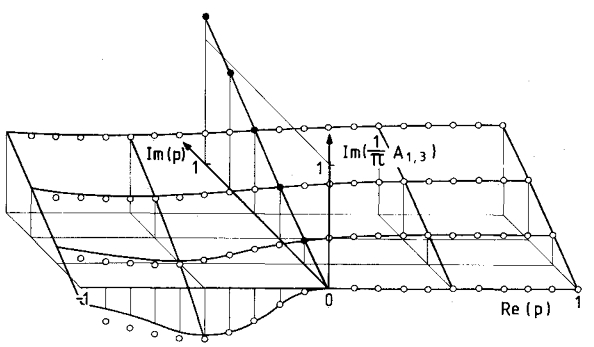

Fig. 5 Comparison of imaginary parts of transfer functions from indicial coefficient method and least squares fit.

routine for solution of a set of linear equations (based on Gaussian elimination and triangularization) for calculation of the determinant.

As the roots shall be determined for increasing values of the flight speed, the approximations required for the roots for each speed are furnished by the roots determined for the previous speed, and the approximations for the first speed are furnished by given eigenvalues for zero speed (measured in a ground vibration test). The procedure has been tested with good results in a practical flutter calculation based on measured data for a relatively small number of modes. Depending on the desired accuracy, only one to three iterations were needed.

The method has the advantage of being simple and applicable without extension or transformation of the matrix. It generates only relevant roots and works even if the root-loci cross each other. At the same speed this happens very seldom.

Conclusions

For calculation of the deflection of an elastic wing, the aerodynamic forces must be expressed in a general way so that arbitrary variation of the deflection with time is permitted. This has been achieved by defining indicial aerodynamic coefficients for elastic modes for a finite wing in subsonic or supersonic linearized flow. By means of these, the aerodynamic forces can be expressed explicitly in terms of the generalized coordinates, which yields equations of motion in the form of integro-differential equations containing convolution integrals.

Laplace transformation of the indicial coefficients gives aerodynamic transfer functions. These are analytic for $|\arg(p)| < \pi$ for $M < 1$ and for $M > 1$ for all finite values of p.

Due to the analyticity, the integration path of the inverse Laplace transform integral can be reduced. Part of it becomes the two sides of the ray $\arg(p) = \pi$. The integral over this part gives a significant contribution to the normal response function in the case of very light wings.

The remaining and usually more important part of the solution is determined by the zeros of the determinant or the eigenvalues of the transformed equations of motion. If a zero lies in the right half-plane, the solution is unstable.

As an alternative to transfer function approximations of the Jones type, which have poles and yield so-called aerodynamic lag roots on $\arg(p) = \pi$, an approximation of the Garrick type, which has a discontinuity like that of the Theodorsen function, has been proposed.

The Newton-Raphson method for solution of a nonlinear equation, combined with efficient numerical methods for calculation of the determinant and its derivative, has been found useful for root-locus generation. This method, which is simple and applicable without increasing the order of the matrix, generates only relevant roots and works even if root-loci cross each other. This happens very seldom for the same speed.

Acknowledgment

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References

[1]Wagner, H., "Über die Entstehung des dynamischen Auftriebes von Tragflügeln," *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 5, No. 1, Feb. 1925, pp. 17-35.

[2]Söhngen, H., "Bestimmung der Auftriebsverteilung für beliebige instationäre Bewegungen (Ebenes Problem)," *Luftfahrtforschung*, Vol. 17, No. 11/12, 1940, pp. 401-420.

[3]Strang, W. J., "Transient Lift of Three-Dimensional Purely Supersonic Wings," *Proceedings of the Royal Society of London*, Vol. A202, No. 1068, June 1950, pp. 54-80.

[4]Drischler, J. A., "Approximate Indicial Lift Functions for Several Wings of Finite Span in Incompressible Flow as Obtained from Oscillatory Lift Coefficients," NACA TN 3639, 1956.

[5]Drischler, J. A. Jr., "Calculation and Compilation of the Unsteady Lift Functions for a Rigid Wing Subjected to Sinusoidal Gusts and to Sinusoidal Sinking Oscillations," NACA TN 3748, Oct. 1956.

[6]Diederich, F. W. and Drischler, J. A. Jr., "Unsteady-Lift Functions for Penetration of Traveling Gusts and Oblique Blast Waves," *Aeronautical Engineering Review*, Aug. 1957, pp. 47-51.

[7]Vogel, U., "Ein Traglinienverfahren zur Berechnung der Auftriebsverteilung von Tragflügeln bei zeitabhängiger Vertikalbewegung," *Zeitschrift für Flugwissenschaften*, Vol. 12, No. 4, 1964, pp. 142-151.

[8]Vepa, R., "On the Use of Padé Approximants to Represent Unsteady Aerodynamic Loads for Arbitrarily Small Motions of Wings," AIAA Paper 76-17, 1976.

[9]Edwards, J. W., "Unsteady Aerodynamic Modeling for Arbitrary Motions," *AIAA Journal*, Vol. 15, April 1977, pp. 593-595.

[10]Stark, V. J. E., "Flutter Calculation for the Viggen Aircraft with Allowance for Leading Edge Vortex Effect," *Unsteady Airloads in Separated and Transonic Flow*, AGARD CP 226, April 1977.

[11]Edwards, J. W., Ashley, H., and Breakwell, J. V., "Unsteady Aerodynamic Modeling for Arbitrary Motions," AIAA Paper 77-451, 1977.

[12]Edwards, J. W., "Applications of Laplace Transform Methods to Airfoil Motion and Stability Calculations," AIAA Paper 79-0772, 1979.

[13]Schwanz, R. C., "A Criterion to Judge the Accuracy of Frequency Domain Approximating Functions," AIAA Paper 80-0767, May 1980.

[14]Garrick, J. E., "On Some Reciprocal Relations in the Theory of Nonstationary Flows," NACA Rept. 629, 1938.

[15]James, H. M. and Weiss, P. R., "Mathematical Background," *Theory of Servomechanisms*, edited by H. M. James, N. B. Nichols, and R. S. Phillips, New York, 1947, pp. 23-75.

[16]Sears, W. R., "Operational Methods in the Theory of Airfoils in Non-Uniform Motion," *Journal of the Franklin Institute*, Vol. 230, No. 1, July 1940, pp. 95-111.

[17]Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., "Aerodynamic Tools: Compressible Flow," *Aeroelasticity*, Cambridge, 1955, pp. 294-379.

[18]Belotserkovskii, S. M., "Calculating the Effect of Gusts on Arbitrary Thin Wing," *Mekhanika Zidkosti i Gaza*, Vol. 1, No. 1, 1967, pp. 34-40 (translation in *Fluid Dynamics*, Vol. 1, No. 1, 1967).

[19]Gautschi, W., and Cahill, W. F., "Exponential Integral and Related Functions," *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun, National Bureau of Standards Applied Mathematics Series 55, 9th printing, 1970, pp. 227-251.

[20]Stark, V. J. E., "A Method for Solving the Subsonic Problem of the Oscillating Finite Wing with the Aid of High-Speed Digital Computers," Saab Aircraft Co., Sweden, Technical Note No. 41, 1958.